A maximum modulus estimate for solutions of the Navier-Stokes system in domains of polyhedral type

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Abstract

The authors prove a maximum modulus estimate for solutions of the stationary Navier-Stokes system in bounded domains of polyhedral type.

Keywords: Navier-Stokes system, nonsmooth domains

MSC (1991): 35J25, 35J55, 35Q30

1 Introduction

The present paper is concerned with solutions of the boundary value problem

$$-\nu\,\Delta v + (v\cdot\nabla)\,v + \nabla p = 0, \quad \nabla\cdot v = 0 \;\text{ in } \Omega, \quad v|_{\partial\Omega} = \phi \tag{1}$$

($\nu > 0$), where Ω is a domain of polyhedral type. This means that the boundary $\partial\Omega$ is the union of a finite number of nonintersecting faces (two-dimensional open manifolds of class C^2), edges (open arcs of class C^2), and vertices (the endpoints of the edges). For every edge point or vertex x_0, there exist a neighborhood U and a diffeomorphism $\kappa : U \to \mathbb{R}^3$ of class C^2 mapping $U\cap\Omega$ onto the intersection of the unit ball with a polyhedron. Note that the results of this paper are also valid for domains of the class Λ^2 introduced in [2].

It is well-known that the solution of the boundary value problem

$$-\Delta w + \nabla q = 0, \quad \nabla\cdot w = 0 \;\text{ in } \Omega, \quad w|_{\partial\Omega} = \phi \tag{2}$$

for the linear Stokes system in a domain $\Omega\subset\mathbb{R}^3$ with smooth boundary $\partial\Omega$ satisfies the estimate

$$\|w\|_{L_\infty(\Omega)} \le c\,\|\phi\|_{L_\infty(\partial\Omega)} \tag{3}$$

with a constant c independent of ϕ. This inequality was first established without proof by Odquist [6]. A proof of this inequality is given e.g. in the book by Ladyzhenskaya. We refer also to the papers of Naumann [5] and Maremonti [1]. Using pointwise estimates of Green's matrix, Maz'ya and Plamenevskiĭ [2] proved the inequality (3) for solutions of problem (2) in domains of polyhedral type.

For the nonlinear problem (1), Solonnikov [7] showed that the solution satisfies the estimate

$$\|v\|_{L_\infty(\Omega)} \le c\,\big(\|\phi\|_{L_\infty(\partial\Omega)}\big), \tag{4}$$

with a certain function c if the boundary $\partial\Omega$ is smooth. Maz'ya and Plamenevskiĭ [2] proved for domains of polyhedral type that the solution v of (1) with finite Dirichlet integral is continuous in $\bar\Omega$ if ϕ is continuous on $\partial\Omega$. However, [2] contains no estimates for the maximum modulus of v. The goal of the present paper is to generalize Solonnikov's result to solutions of problem (1) in domains of polyhedral type and to obtain a more precise estimate. The function c constructed in the present paper has the form

$$c(t) = c_0 t\, e^{c_1 t/\nu}, \tag{5}$$

where c_0 and c_1 are positive constants independent of ν.

2 Estimates for solutions of the linear Stokes system

First, we consider problem (2). Throughout this paper, we assume that $\phi \in L_\infty(\partial\Omega)$ and

$$\int_{\partial\Omega} \phi \cdot n \, d\sigma = 0. \tag{6}$$

The following two lemmas were proved in [7] for domains with smooth boundaries. We give here other proofs which do not require the smoothness of the boundary $\partial\Omega$. In particular for the proof of Lemma 1, we will employ the estimates of Green's matrix given in [2].

Lemma 1 *Let Ω be a domain of polyhedral type, and let (w, q) be the solution of problem (2) satisfying the condition $\int_\Omega q(x)\, dx = 0$. Then there exists a constant c independent of ϕ such that the inequalities (3) and*

$$\sup_{x\in\Omega} d(x) \Big(\sum_{j=1}^{3} \big|\partial_{x_j} w(x)\big| + \big|q(x)\big| \Big) \le c \, \|\phi\|_{L_\infty(\partial\Omega)} \tag{7}$$

are satisfied, where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

P r o o f. As mentioned in the introduction, the inequality (3) was proved in [2, Cor.9.2]. We include its proof for readers' convenience. Let $G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$ denote the Green matrix for problem (2). This means that the vectors $\vec{G}_j = (G_{1,j}, G_{2,j}, G_{3,j})$ and the function $G_{4,j}$ are the uniquely determined solutions of the problems

$$\begin{aligned}
&-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\, \vec{e}_j, \quad \nabla_x \cdot \vec{G}_j(x,\xi) = 0 \quad \text{for } x,\xi \in \Omega,\ j = 1,2,3,\\
&-\Delta_x \vec{G}_4(x,\xi) + \nabla_x G_{4,4}(x,\xi) = 0, \quad \nabla_x \cdot \vec{G}_4(x,\xi) = \delta(x-\xi) - (\operatorname{mes}(\Omega))^{-1} \quad \text{for } x,\xi \in \Omega,\\
&\vec{G}_j(x,\xi) = 0 \ \text{ for } x \in \partial\Omega,\ \xi \in \Omega,\ j = 1,2,3,4,
\end{aligned}$$

satisfying the condition

$$\int_\Omega G_{4,j}(x,\xi)\, dx = 0 \text{ for } \xi \in \Omega,\ j = 1,2,3,4.$$

Here $\vec{e}_j$ denotes the vector $(\delta_{1,j}, \delta_{2,j}, \delta_{3,j})$. Then the components of the vector function w and q have the representation

$$\begin{aligned}
w_i(x) &= \int_{\partial\Omega} \Big(-\sum_{j=1}^{3} \frac{\partial G_{i,j}(x,\xi)}{\partial n_\xi}\, \phi_j(\xi) + G_{i,4}(x,\xi)\, \phi(\xi)\cdot n_\xi \Big)\, d\sigma_\xi,\ i = 1,2,3,\\
q(x) &= \int_{\partial\Omega} \Big(-\sum_{j=1}^{3} \frac{\partial G_{4,j}(x,\xi)}{\partial n_\xi}\, \phi_j(\xi) + G_{4,4}(x,\xi)\, \phi(\xi)\cdot n_\xi \Big)\, d\sigma_\xi.
\end{aligned}$$

For the proof of (3) and (7), we employ the estimates of the functions $G_{i,j}$ given in [2] (for a more general boundary value problem in a cone with edges see also [3]). We start with the inequality (7). Suppose that x lies in a neighborhood $\mathcal{U}$ of the vertex $x^{(1)}$. We denote by $\mathcal{S}$ the set of the vertices and edge points of the boundary $\partial\Omega$, by $\rho_i(x)$ the distance of x from the vertex $x^{(i)}$, by $r_k(x)$ the distance from the edge M_k, by $r(x) = \min_k r_k(x)$ the distance from the set of all edge points, and introduce the following subsets of $\mathcal{U} \cap (\partial\Omega\backslash\mathcal{S})$:

$$\begin{aligned}
E_1 &= \{\xi \in \mathcal{U} \cap (\partial\Omega\backslash\mathcal{S}) :\ \rho_1(\xi) > 2\rho_1(x)\},\\
E_2 &= \{\xi \in \mathcal{U} \cap (\partial\Omega\backslash\mathcal{S}) :\ \rho_1(\xi) < \rho_1(x)/2\},\\
E_3 &= \{\xi \in \mathcal{U} \cap (\partial\Omega\backslash\mathcal{S}) :\ \rho_1(x)/2 < \rho_1(\xi) < 2\rho_1(x),\ |x-\xi| > \min(r(x), r(\xi))\},\\
E_4 &= \{\xi \in \mathcal{U} \cap (\partial\Omega\backslash\mathcal{S}) :\ \rho_1(x)/2 < \rho_1(\xi) < 2\rho_1(x),\ |x-\xi| < \min(r(x), r(\xi))\}.
\end{aligned}$$

Let $K(x,\xi)$ be one of the functions

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial n_\xi} G_{i,j}(x,\xi), \quad \frac{\partial G_{i,4}(x,\xi)}{\partial x_j}, \quad \frac{\partial}{\partial n_\xi} G_{4,j}(x,\xi), \quad G_{4,4}(x,\xi),$$

$i,j=1,2,3$. Then the following estimates are valid for $x\in\mathcal{U},\ \xi\in\mathcal{U}\cap(\partial\Omega\backslash\mathcal{S})$:

$$\begin{aligned}
\big|K(x,\xi)\big| &\le c\,\rho_1(x)^{\Lambda-1}\,\rho_1(\xi)^{-\Lambda-2}\prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k-1}\prod_{k\in J_1}\Big(\frac{r_k(\xi)}{\rho_1(\xi)}\Big)^{\mu_k-1} \quad \text{for } \xi\in E_1,\\
\big|K(x,\xi)\big| &\le c\,\rho_1(x)^{-\Lambda-2}\,\rho_1(\xi)^{\Lambda-1}\prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k-1}\prod_{k\in J_1}\Big(\frac{r_k(\xi)}{\rho_1(\xi)}\Big)^{\mu_k-1} \quad \text{for } \xi\in E_2,\\
\big|K(x,\xi)\big| &\le c\,|x-\xi|^{-3}\Big(\frac{r(x)}{|x-\xi|}\Big)^{\mu-1}\Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\mu-1} \quad \text{for } \xi\in E_3,\\
\big|K(x,\xi)\big| &\le c\,|x-\xi|^{-3} \quad \text{for } \xi\in E_4,
\end{aligned}$$

where $\Lambda>0$, $\mu_k>1/2$, $\mu>1/2$. Here J_l is the set of all indices k such that $x^{(l)}\in\overline{M}_k$. Note that

$$c_1\,r(x)\le\rho_1(x)\prod_{k\in J_1}\frac{r_k(x)}{\rho_1(x)}\le c_2\,r(x)\quad \text{for } x\in\mathcal{U},$$

where c_1 and c_2 are positive constants. We consider the integral

$$I(x)=\int\limits_{\partial\Omega\cap\mathcal{U}}K(x,\xi)\,\psi(\xi)\,d\sigma_\xi$$

for $x\in\mathcal{U}$, $\psi\in L_\infty(\partial\Omega)$ and write this integral as a sum $I(x)=I_1+I_2+I_3+I_4$, where I_k is the integral of $K(x,\xi)\,\psi(\xi)$ over the set E_k, $k=1,2,3,4$. Then

$$\begin{aligned}
I_1 &\le c\,\rho_1(x)^{\Lambda-1}\prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k-1}\|\psi\|_{L_\infty(\partial\Omega)}\int_{E_1}\rho_1(\xi)^{-\Lambda-2}\prod_{k\in J_1}\Big(\frac{r_k(\xi)}{\rho_1(\xi)}\Big)^{\mu_k-1}d\sigma_\xi\\
&\le c\,\rho_1(x)^{-1}\prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k-1}\|\psi\|_{L_\infty(\partial\Omega)}\le c\,r(x)^{-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\,.
\end{aligned}$$

Analogously, the inequality

$$I_2\le c\,r(x)^{-1}\,\|\psi\|_{L_\infty(\partial\Omega)}$$

holds. Suppose without loss of generality that M_1 is the nearest edge to x. We denote by $E_3^{(1)}$ the set of all $\xi\in E_3$ such that $r(\xi)<r_1(\xi)$. Furthermore, let $I_3^{(1)}$ be the integral of $K(x,\xi)\,\psi(\xi)$ over the set $E_3^{(1)}$. If $\xi\in E_3^{(1)}$, then there exists a positive constant c such that $|x-\xi|>c\,\rho_1(x)$. Hence

$$I_3^{(1)}\le c\,\rho_1(x)^{-2\mu-1}\,r_1(x)^{\mu-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\int_{E_3^{(1)}}r(\xi)^{\mu-1}\,d\sigma_\xi.$$

Since $E_3^{(1)}\subset\{\xi:\ \rho_1(x)/2<\rho_1(\xi)<2\rho_1(x)\}$ and $r_1(x)\le\rho_1(x)$, we obtain

$$I_3^{(1)}\le c\,\rho_1(x)^{-\mu}r_1(x)^{\mu-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\le c\,r_1(x)^{-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\,.$$

Let $\xi\in E_3\backslash E_3^{(1)}$ and let x', ξ' denote the nearest points on the edge M_1 to x and ξ, respectively. Then there exists a positive constant c independent of x and ξ such that

$$|x-\xi|>c\,\big(r(x)+r(\xi)+|x'-\xi'|\big).$$

Consequently,

$$\begin{aligned}
|I_3-I_3^{(1)}| &\le c\,r(x)^{\mu-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\int\limits_{E_3\backslash E_3^{(1)}}\frac{r(\xi)^{\mu-1}}{(r(x)+r(\xi)+|x'-\xi'|)^{2\mu+1}}\,d\sigma_\xi\\
&\le c\,r(x)^{\mu-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\int_0^\infty\int_{\mathbb{R}}\frac{r^{\mu-1}}{(r+r(x)+|t|)^{2\mu+1}}\,d\xi'\,dr=C\,r(x)^{-1}\,\|\psi\|_{L_\infty(\partial\Omega)}\,.
\end{aligned}$$

Finally using the estimate for $K(x,\xi)$ in E_4, we obtain

$$I_4 \le c\,\|\psi\|_{L_\infty(\partial\Omega)} \int_{E_4} |x-\xi|^{-3}\, d\sigma_\xi \le C\, d(x)^{-1} \|\psi\|_{L_\infty(\partial\Omega)}\,.$$

Thus we have shown that

$$I(x) \le c\, d(x)^{-1}\, \|\psi\|_{L_\infty(\partial\Omega)} \quad \text{for } x\in \Omega\cap\mathcal{U}.$$

Now, we consider the integral

$$\int\limits_{\partial\Omega\cap\mathcal{V}} K(x,\xi)\,\psi(\xi)\, d\sigma_\xi \tag{8}$$

for $x\in\Omega\cap\mathcal{U}$, where $\mathcal{V}$ is a neighborhood of the vertex $x^{(l)}$, $l\neq 1$. Using the estimate

$$\big|K(x,\xi)\big| \le c\,\rho_1(x)^{\Lambda-1}\,\rho_l(\xi)^{\Lambda-1} \prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k-1} \prod_{k\in J_l}\Big(\frac{r_k(\xi)}{\rho_l(\xi)}\Big)^{\mu_k-1} \quad \text{for } x\in\mathcal{U},\ \xi\in\mathcal{V},$$

we obtain

$$\Big|\int\limits_{\partial\Omega\cap\mathcal{V}} K(x,\xi)\,\psi(\xi)\,d\sigma_\xi\Big| \le c\,\rho_1(x)^{\Lambda-1} \prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k-1} \|\psi\|_{L_\infty(\partial\Omega)} \le c\, r(x)^{-1}\, \|\psi\|_{L_\infty(\partial\Omega)}\,.$$

The same estimate holds for the integral (8) in the case when $\mathcal{V}$ is a neighborhood of an arbitrary other boundary point. This proves (7). Analogously, (3) holds by means of the estimates

$$\begin{aligned}
\big|K(x,\xi)\big| &\le c\,\rho_1(x)^{\Lambda}\,\rho_1(\xi)^{-\Lambda-2} \prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k} \prod_{k\in J_1}\Big(\frac{r_k(\xi)}{\rho_1(\xi)}\Big)^{\mu_k-1} \quad \text{for } \xi\in E_1,\\
\big|K(x,\xi)\big| &\le c\,\rho_1(x)^{-\Lambda-1}\,\rho_1(\xi)^{\Lambda-1} \prod_{k\in J_1}\Big(\frac{r_k(x)}{\rho_1(x)}\Big)^{\mu_k} \prod_{k\in J_1}\Big(\frac{r_k(\xi)}{\rho_1(\xi)}\Big)^{\mu_k-1} \quad \text{for } \xi\in E_2,\\
\big|K(x,\xi)\big| &\le c\,|x-\xi|^{-2}\Big(\frac{r(x)}{|x-\xi|}\Big)^{\mu}\Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\mu-1} \quad \text{for } \xi\in E_3,\\
\big|K(x,\xi)\big| &\le c\,d(x)\,|x-\xi|^{-3} \quad \text{for } \xi\in E_4,
\end{aligned}$$

for the functions $K(x,\xi)=\partial G_{i,j}(x,\xi)/\partial n_\xi$ and $K(x,\xi)=G_{i,4}(x,\xi)$, $i,j=1,2,3$ (see [2, Th.9.1]). $\square$

We denote by $W^{l,p}(\Omega)$ the Sobolev space with the norm

$$\|u\|_{W^{l,p}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha|\le l} \big|\partial_x^\alpha u(x)\big|^p\, dx\Big)^{1/p}.$$

Here l is a nonnegative integer and $1<p<\infty$.

Lemma 2 *Let (w,q) be a solution of problem (2), where Ω is a domain of polyhedral type. Then there exists a vector function $b\in W^{1,6}(\Omega)^3$ such that $w=\operatorname{rot} b$ and*

$$\|b\|_{W^{1,6}(\Omega)} \le c\,\|\phi\|_{L_\infty(\partial\Omega)} \tag{9}$$

with a constant c independent of ϕ.

P r o o f. Let B_ρ be a ball with radius ρ centered at the origin and such that $\overline{\Omega}\subset B_\rho$. Furthermore, let $(w^{(1)},s)$ be a solution of the problem

$$-\Delta w^{(1)} + \nabla s = 0, \quad \nabla\cdot w^{(1)} = 0 \text{ in } B_\rho\backslash\overline{\Omega}, \quad w^{(1)}|_{\partial\Omega}=\phi, \quad w^{(1)}|_{\partial B_\rho}=0.$$

Obviously, the vector function

$$u(x) = \begin{cases} w(x) & \text{for } x\in\Omega,\\ w^{(1)}(x) & \text{for } x\in B_\rho\backslash\Omega \end{cases}$$

satisfies the equality $\nabla \cdot u = 0$ in the sense of distributions in B_ρ. Due to Lemma 1, the L_∞ norms of w and $w^{(1)}$ can be estimated by the L_∞ norm of ϕ. Hence,

$$\|u\|_{L_6(B_\rho)} \le c\, \|\phi\|_{L_\infty(\partial\Omega)}\,,$$

where c is a constant independent of ϕ. Suppose that there exists a vector function $U \in W^{2,6}(B_\rho)^3$ satisfying the equations

$$-\Delta U = u \ \text{ in } B_\rho, \quad \nabla \cdot U = 0 \ \text{ on } \partial B_\rho \tag{10}$$

and the inequality

$$\|U\|_{W^{2,6}(B_\rho)^3} \le c\, \|u\|_{L_6(B_\rho)^3}\,. \tag{11}$$

Since $\Delta(\nabla \cdot U) = \nabla \cdot u = 0$ in B_ρ it follows that $\nabla \cdot U = 0$ in B_ρ. Consequently for the vector function $b = \operatorname{rot} U$, we obtain

$$\operatorname{rot} b = \operatorname{rot}\operatorname{rot} U = -\Delta U + \operatorname{grad}\operatorname{div} U = u \ \text{ in } B_\rho$$

and

$$\|b\|_{W^{1,6}(B_\rho)^3} \le c_1\, \|U\|_{W^{2,6}(B_\rho)^3} \le c\, c_1\, \|u\|_{L_6(B_\rho)^3} \le c_2\, \|\phi\|_{L_\infty(\partial\Omega)}\,.$$

It remains to show that problem (10) has a solution U subject to (11). To this end, we consider the boundary value problem

$$-\Delta U = u \ \text{ in } B_\rho, \quad \frac{\partial U_r}{\partial r} + \frac{2}{r}\, U_r = U_\theta = U_\varphi = 0 \ \text{ on } \partial B_\rho, \tag{12}$$

where U_r, U_θ, U_φ are the spherical components of the vector function U, i.e.

$$\begin{pmatrix} U_r \\ U_\theta \\ U_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\, \cos\varphi & \sin\theta\, \sin\varphi & \cos\theta \\ \cos\theta\, \cos\varphi & \cos\theta\, \sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}.$$

On the set of all U satisfying the boundary conditions in (12), we have

$$\begin{aligned} -\int_{B_\rho} \Delta U \cdot \bar U\, dx &= \sum_{j=1}^3 \int_{B_\rho} \left|\partial_{x_j} U\right|^2 dx - \rho^{-1} \int_{\partial B_\rho} \frac{\partial U}{\partial r} \cdot \bar U\, d\sigma \\ &= \sum_{j=1}^3 \int_{B_\rho} \left|\partial_{x_j} U\right|^2 dx - \rho^{-1} \int_{\partial B_\rho} \frac{\partial U_r}{\partial r} \cdot \bar U_r\, d\sigma = \sum_{j=1}^3 \int_{B_\rho} \left|\partial_{x_j} U\right|^2 dx + 2\rho^{-2} \int_{\partial B_\rho} |U_r|^2\, d\sigma. \end{aligned}$$

Since the quadratic form on the right-hand side is coercive, problem (12) is uniquely solvable in $W^{1,2}(B_\rho)^3$. By a well-known regularity result for solutions of elliptic boundary value problems, the solution belongs to $W^{2,6}(B_\rho)^3$ and satisfies (11) if $u \in L_6(B_\rho)^3$. From (12) and from the equality

$$\nabla \cdot U = \frac{\partial U_r}{\partial r} + \frac{2}{r}\, U_r + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{\cot\theta}{r}\, U_\theta + \frac{1}{r \sin\theta}\, \frac{\partial U_\varphi}{\partial \varphi}$$

it follows that $\nabla \cdot U = 0$ on ∂B_ρ. The proof of the lemma is complete. □

Next, we consider the solution (W, Q) of the problem

$$-\Delta W + \nabla Q = f, \quad \nabla \cdot W = 0 \ \text{ in } \Omega, \quad W|_{\partial\Omega} = 0. \tag{13}$$

Suppose that $x^{(1)}, \dots, x^{(d)}$ are the vertices and $M_1, \dots, M_m$ the edges of Ω. As in the proof of Lemma 1, we use the notation $\rho_j(x) = \operatorname{dist}(x, x^{(j)})$, $r_k(x) = \operatorname{dist}(x, M_k)$, $\rho(x) = \min_j \rho_j(x)$, and $r(x) = \min_k r_k(x)$. Then $V^{l,s}_{\beta,\delta}(\Omega)$ is defined as the weighted Sobolev space with the norm

$$\|u\|_{V^{l,s}_{\beta,\delta}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha| \le l} r(x)^{s(|\alpha| - m)} \prod_{j=1}^d \rho_j^{s\beta_j} \ \prod_{k=1}^m \Big(\frac{r_k}{\rho}\Big)^{s\delta_k} \ \left|\partial_x^\alpha u(x)\right|^s dx \Big)^{1/s}.$$

Here, l is a nonnegative integer, $s \in (1, \infty)$, $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$, and $\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$. The space $V^{-1,s}_{\beta,\delta}(\Omega)$ is the set of all distributions of the form $u = u_0 + \nabla \cdot u^{(1)}$, where $u_0 \in V^{0,s}_{\beta+1,\delta+1}(\Omega)$ and $u^{(1)} \in V^{0,s}_{\beta,\delta}(\Omega)^3$. By Theorem [2, Th.6.1] (for a more general boundary value problem see also [4]), problem (13) is uniquely solvable (up to vector functions of the form $(0, c)$, where c is a constant) in $V^{1,s}_{\beta,\delta}(\Omega)^3 \times V^{0,s}_{\beta,\delta}(\Omega)$ for arbitrary $f \in V^{-1,s}_{\beta,\delta}(\Omega)^3$ if

$$|\beta_j - 3/2 + 3/s| < \varepsilon_j + 1/2 \quad \text{and} \quad |\delta_k - 1 + 2/s| < \varepsilon'_k + 1/2\,.$$

Here ε_j and ε'_k are positive numbers depending on Ω. In particular, problem (13) has a unique (up to constant Q) solution $(W, Q) \in V^{1,s}_{0,0}(\Omega)^3 \times V^{0,s}_{0,0}(\Omega)$ satisfying the estimate

$$\|W\|_{V^{1,s}_{0,0}(\Omega)} \le c\, \|f\|_{V^{1,s}_{0,0}(\Omega)} \tag{14}$$

for arbitrary $f \in V^{-1,s}_{0,0}(\Omega)^3$ if $1 < s < 3 + \varepsilon$ with a certain $\varepsilon > 0$. The components of the vector function W admit the representation

$$W_i(x) = \int_\Omega \sum_{j=1}^{3} G_{i,j}(x, \xi)\, f_j(\xi)\, d\xi, \tag{15}$$

where $G_{i,j}(x, \xi)$ are the elements of Green's matrix introduced in the proof of Lemma 1. From (14), we obtain the following estimates.

Lemma 3 *Suppose that $f = \partial_{x_j} g$, where $j \in \{1, 2, 3\}$. If $g \in L_s(\Omega)^3$, $s > 3$, then*

$$\|W\|_{L_\infty(\Omega)} \le c\, \|g\|_{L_s(\Omega)}\,. \tag{16}$$

If $g \in L_3(\Omega)^3$, then

$$\|W\|_{L_s(\Omega)} \le c\, \|g\|_{L_3(\Omega)} \tag{17}$$

for arbitrary s, $1 < s < \infty$.

P r o o f. Let $g \in L_s(\Omega)$, $s > 3$, and let ε be a sufficiently small positive number, $\varepsilon < s - 3$. Then it follows from (14) and from the continuity of the imbeddings $V^{1,3+\varepsilon}_{0,0}(\Omega) \subset W^{1,3+\varepsilon}(\Omega) \subset L_\infty(\Omega)$ that

$$\|W\|_{L_\infty(\Omega)} \le c_1\, \|W\|_{W^{1,3+\varepsilon}(\Omega)} \le c_2\, \|W\|_{V^{1,3+\varepsilon}_{0,0}(\Omega)} \le c_3\, \|g\|_{L_{3+\varepsilon}(\Omega)} \le c_4\, \|g\|_{L_s(\Omega)}\,.$$

Analogously, we obtain

$$\|W\|_{L_s(\Omega)} \le c_5\, \|W\|_{W^{1,3}(\Omega)} \le c_6\, \|W\|_{V^{1,3}_{0,0}(\Omega)} \le c_7\, \|g\|_{L_3(\Omega)}\,.$$

The lemma is proved. □

3 An estimate of the maximum modulus of the solution to the Navier-Stokes system

Now we prove the main result of this paper introducing some modifications into Solonnikov's scheme.

Theorem 1 *Let (v, q) be a solution of problem* (1), *where Ω is a domain of polyhedral type. Then v satisfies the estimate* (4) *with a function c of the form* (5).

P r o o f. Suppose first that $\nu = 1$. Let (w, q) be the solution of problem (2), $\int_\Omega q(x)\, dx = 0$. Then the vector function $(v - w, p - q)$ satisfies the equations

$$-\Delta(v - w) + \nabla(p - q) = -(v \cdot \nabla)\, v, \quad \nabla \cdot (v - w) = 0$$

in Ω and the boundary condition $v - w = 0$ on $\partial\Omega$. Hence by (15), we have $v = w + W$, where W is the vector function with the components

$$W_i(x) = -\int_\Omega \sum_{j=1}^{3} G_{i,j}(x,\xi)\,\big(v(\xi)\cdot\nabla\big)\,v_j(\xi)\,d\xi = -\int_\Omega \sum_{j=1}^{3} G_{i,j}(x,\xi)\,\nabla\cdot\big(v_j(\xi)\,v(\xi)\big)\,d\xi,$$

$i = 1, 2, 3$. Using (16), we obtain

$$\begin{aligned} \|v\|_{L_\infty(\Omega)} &\le \|w\|_{L_\infty(\Omega)} + \|W\|_{L_\infty(\Omega)} \le \|w\|_{L_\infty(\Omega)} + c \sum_{i,j=1}^{3} \|v_i\,v_j\|_{L_{s/2}(\Omega)} \\ &\le \|w\|_{L_\infty(\Omega)} + c\,\|v\|^2_{L_s(\Omega)} \end{aligned} \tag{18}$$

for arbitrary $s > 6$. From (17) it follows that

$$\begin{aligned} \|v\|_{L_s(\Omega)} &\le \|w\|_{L_s(\Omega)} + \|W\|_{L_s(\Omega)} \le \|w\|_{L_s(\Omega)} + c \sum_{i,j=1}^{3} \|v_i\,v_j\|_{L_3(\Omega)} \\ &\le c_1\,\|w\|_{L_\infty(\Omega)} + c_2\,\|v\|^2_{L_6(\Omega)}\,. \end{aligned} \tag{19}$$

Combining (3), (18) and (19), we obtain

$$\|v\|_{L_\infty(\Omega)} \le c_3 \Big(\|\phi\|_{L_\infty(\partial\Omega)} + \|\phi\|^2_{L_\infty(\partial\Omega)} + \|v\|^4_{L_6(\Omega)} \Big). \tag{20}$$

with a certain constant c_3 independent of ϕ.

The norm of v in $L_6(\Omega)$ can be estimated in the same way as in [7]. Let $\delta(x)$ be the regularized distance of x from the boundary $\partial\Omega$ (see [8, Ch.6,§2]), i.e. δ is an infinitely differentiable function on Ω satisfying the inequalities

$$c_1\,d(x) \le \delta(x) \le c_2\,d(x), \qquad \big|\partial_x^\alpha \delta(x)\big| \le c_\alpha\,d(x)^{1-|\alpha|}$$

with certain positive constants c_1, c_2, c_α. Furthermore, let ρ and κ be positive numbers, and let χ be an infinitely differentiable function such that $0 \le \chi \le 1$, $\chi(t) = 0$ for $t \le 0$, and $\chi(t) = 1$ for $t \ge 1$. We define the cut-off function ζ on Ω by

$$\zeta(x) = \chi\Big(\kappa\,\log\frac{\rho}{\delta(x)}\Big).$$

This function has the following properties.

(i) $0 \le \zeta(x) \le 1$, $\zeta(x) = 0$ for $\delta(x) \ge \rho$, $\zeta(x) = 1$ for $\delta(x) \le \varepsilon\rho$, where $\varepsilon = e^{-1/\kappa}$.

(ii) $|\nabla\zeta(x)| \le c\,\dfrac{\kappa}{d(x)}$, $\quad \big|\partial_{x_i}\partial_{x_j}\zeta(x)\big| \le c\dfrac{\kappa}{d(x)^2}$ for $i, j = 1, 2, 3$.

By Lemma 2, the vector function w admits the representation $w = \operatorname{rot} b$ with a vector function $b \in W^{1,6}(\Omega)^3$ satisfying (9). We put

$$v = V + u, \quad \text{where} \quad V = \operatorname{rot}(\zeta b) = \zeta w + \nabla\zeta \times b.$$

Then u satisfies the equations

$$-\Delta u + (v\cdot\nabla)\,u + (u\cdot\nabla)\,V = \Delta V - (V\cdot\nabla)\,V - \nabla p, \quad \nabla\cdot u = 0 \tag{21}$$

in Ω and the boundary condition $u|_{\partial\Omega} = 0$. Since

$$\int_\Omega \big((v\cdot\nabla)\,u\big)\cdot u\,dx = 0,$$

it follows from (21) that

$$\sum_{j=1}^{3} \|\nabla u_j\|^2_{L_2(\Omega)} - \sum_{j=1}^{3} \int_\Omega u_j\, V \cdot \frac{\partial u}{\partial x_j}\, dx = L(u), \tag{22}$$

where

$$\begin{aligned} L(u) &= \int_\Omega \Big(\Delta V - (V\cdot\nabla)V - \nabla p\Big)\cdot u\, dx = \sum_{j=1}^{3}\int_\Omega \Big(-\nabla V_j \cdot \nabla u_j + V_j\, V \cdot \frac{\partial u}{\partial x_j}\Big)\, dx \\ &= -\int_\Omega \Big(w\cdot u\,\Delta\zeta + 2w\cdot(\nabla\zeta\cdot\nabla)\,u + q\,u\cdot\nabla\zeta\Big)\, dx - \sum_{j=1}^{3}\int_\Omega \nabla(\nabla\zeta\times b)_j\cdot\nabla u_j\, dx \\ &\quad + \sum_{j=1}^{3}\int_\Omega V_j\, V\cdot\frac{\partial u}{\partial x_j}\, dx \end{aligned}$$

(here $(\nabla\zeta\times b)_j$ denotes the jth component of the vector $\nabla\zeta\times b$). We estimate the functional $L(u)$. Using the inequalities

$$|\nabla\zeta| \le c\,\frac{\kappa}{\varepsilon\rho}\,, \qquad \big|d\,\Delta\zeta\big| \le c\,\frac{\kappa}{\varepsilon\rho}\,,$$

$$\int_\Omega d(x)^{-2}\,\big|u(x)\big|^2\, dx \le c\int_\Omega \big|\nabla u(x)\big|^2\, dx$$

(the last follows from Hardy's inequality) and (3), we obtain

$$\Big|\int_\Omega w\cdot u\,\Delta\zeta\, dx\Big| \le \|w\|_{L_\infty(\Omega)}\,\|d\Delta\zeta\|_{L_2(\Omega)}\,\|d^{-1}u\|_{L_2(\Omega)} \le c\,\frac{\kappa}{\varepsilon\rho}\,\|\phi\|_{L_\infty(\partial\Omega)}\sum_{j=1}^{3}\|\nabla u_j\|_{L_2(\Omega)}$$

and

$$\Big|\int_\Omega w\cdot(\nabla\zeta\cdot\nabla)\,u\, dx\Big| \le \|w\|_{L_\infty(\Omega)}\,\|\nabla\zeta\|_{L_2(\Omega)}\sum_{j=1}^{3}\|\nabla u_j\|_{L_2(\Omega)} \le c\,\frac{\kappa}{\varepsilon\rho}\,\|\phi\|_{L_\infty(\partial\Omega)}\sum_{j=1}^{3}\|\nabla u_j\|_{L_2(\Omega)}\,.$$

Analogously by (7) and (9),

$$\Big|\int_\Omega q\,u\cdot\nabla\zeta\, dx\Big| \le \|qd\|_{L_\infty(\Omega)}\,\|d^{-1}u\|_{L_2(\Omega)}\,\|\nabla\zeta\|_{L_2(\Omega)} \le c\,\frac{\kappa}{\varepsilon\rho}\,\|\phi\|_{L_\infty(\partial\Omega)}\sum_{j=1}^{3}\|\nabla u_j\|_{L_2(\Omega)}\,,$$

$$\begin{aligned} \Big|\int_\Omega \nabla(\nabla\zeta\times b)_j\cdot\nabla u_j\, dx\Big| &\le c\Big(\|\nabla\zeta\|_{L_\infty(\Omega)}\,\|\nabla b\|_{L_2(\Omega)} + \sum_{i,k}\|\frac{\partial^2\zeta}{\partial x_i\partial x_k}\|_{L_\infty(\Omega)}\,\|b\|_{L_2(\Omega)}\Big)\,\|\nabla u_j\|_{L_2(\Omega)} \\ &\le c\,\frac{\kappa}{\varepsilon^2\rho^2}\,\|\phi\|_{L_\infty(\partial\Omega)}\,\|\nabla u_j\|_{L_2(\Omega)}\,, \end{aligned}$$

and

$$\begin{aligned} \Big|\int_\Omega V_j\,V\cdot\frac{\partial u}{\partial x_j}\, dx\Big| &\le \|V\|^2_{L_4(\Omega)}\,\|\partial_{x_j}u\|_{L_2(\Omega)} \le 2\Big(\|\zeta w\|^2_{L_4(\Omega)} + \|\nabla\zeta\times b\|^2_{L_4(\Omega)}\Big)\,\|\partial_{x_j}u\|_{L_2(\Omega} \\ &\le c\Big(1+\frac{\kappa^2}{\varepsilon^2\rho^2}\Big)\,\|\phi\|^2_{L_2(\Omega)}\,\|\partial_{x_j}u\|_{L_2(\Omega)}\,. \end{aligned}$$

Thus,

$$\big|L(u)\big| \le C_1\Big(\frac{\kappa}{\varepsilon^2\rho^2}\,\|\phi\|_{L_\infty(\partial\Omega)} + \big(1+\frac{\kappa^2}{\varepsilon^2\rho^2}\big)\,\|\phi\|^2_{L_\infty(\partial\Omega)}\Big)\,\|\nabla u\|_{L_2(\Omega)}\,, \tag{23}$$

where C_1 is a constant independent of ρ and κ. Furthermore,

$$\Big|\sum_{j=1}^{3}\int_\Omega u_j\, V\cdot\frac{\partial u}{\partial x_j}\,dx\Big| = \Big|\sum_{j=1}^{3}\int_\Omega u_j(\zeta w+\nabla\zeta\times b)\cdot\frac{\partial u}{\partial x_j}\,dx\Big|$$

$$\leq \Big(\|\zeta d\|_{L_\infty(\Omega)}\,\|w\|_{L_\infty(\Omega)}+\|d\,\nabla\zeta\|_{L_\infty(\Omega)}\,\|b\|_{L_\infty(\Omega)}\Big)\sum_{j=1}^{3}\|d^{-1}u_j\|_{L_2(\Omega)}\,\|\partial_{x_j}u\|_{L_2(\Omega)}$$

$$\leq C_2\,(\rho+\kappa)\,\|\phi\|_{L_\infty(\Omega)}\sum_{j=1}^{3}\|\nabla u_j\|_{L_2(\Omega)}\,,$$

where C_2 is independent of ϕ,ρ,κ. The numbers ρ and κ can be chosen such that

$$C_2\,(\rho+\kappa)\,\|\phi\|_{L_\infty(\partial\Omega)}\leq 1/2.$$

Then it follows from (22) and (23) that

$$\sum_{j=1}^{3}\|\nabla u_j\|_{L_2(\Omega)}\leq 2\,C_1\left(\frac{\kappa}{\varepsilon^2\rho^2}\,\|\phi\|_{L_\infty(\partial\Omega)}+\big(1+\frac{\kappa^2}{\varepsilon^2\rho^2}\big)\,\|\phi\|^2_{L_\infty(\partial\Omega)}\right).$$

By the continuity of the imbedding $W^{1,2}(\Omega)\subset L_6(\Omega)$, the same estimate (with another constant C_1) holds for the norm of u in $L_6(\Omega)^3$. Since $|\nabla\zeta|\leq c\kappa/(\varepsilon\rho)$, we further have

$$\|V\|_{L_6(\Omega)}\leq\|\zeta w\|_{L_6(\Omega)}+\|\nabla\zeta\times b\|_{L_6(\Omega)}\leq C_3\,\big(1+\kappa/(\varepsilon\rho)\big)\,\|\phi\|_{L_\infty(\partial\Omega)} \tag{24}$$

(see Lemmas 1 and 2) and consequently

$$\|v\|_{L_6(\Omega)}\leq\|V\|_{L_6(\Omega)}+\|u\|_{L_6(\Omega)}\leq C_4\left(\big(1+\frac{\kappa}{\varepsilon\rho}+\frac{\kappa}{\varepsilon^2\rho^2}\big)\,\|\phi\|_{L_\infty(\partial\Omega)}+\big(1+\frac{\kappa^2}{\varepsilon^2\rho^2}\big)\,\|\phi\|^2_{L_\infty(\partial\Omega)}\right).$$

If we put

$$\kappa=\rho=\frac{1}{4C_2\,\|\phi\|_{L_\infty(\partial\Omega)}}\quad\text{and}\quad\varepsilon=e^{-1/\kappa}=e^{-4C_2\|\phi\|_{L_\infty(\partial\Omega)}}\,,$$

we obtain

$$\|v\|_{L_6(\Omega)}\leq C_5\left(\|\phi\|_{L_\infty(\partial\Omega)}\,e^{4C_2\|\phi\|_{L_\infty(\partial\Omega)}}+\|\phi\|^2_{L_\infty(\partial\Omega)}\,e^{8C_2\|\phi\|_{L_\infty(\partial\Omega)}}\right).$$

This together with (20) implies (4) for $\nu=1$. If $\nu\neq 1$, then we consider the vector function $(\nu^{-1}v,\nu^{-2}p)$ instead of (v,p). $\square$

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